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Convergence of the gaussian expansion method in dimensionally reduced Yang-Mills integrals

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Abstract: We advocate a method to improve systematically the self-consistent harmonic approximation (or the gaussian approximation), which has been employed extensively in condensed matter physics and statistical mechanics. Such a method was previously applied to the IIB matrix model, a conjectured nonperturbative definition of type-IIB superstring theory in ten dimensions. Remarkably the dominance of four-dimensional space-time in the partition function was suggested from calculations up to the 3rd order. Recently this calculation has been extended to the 5th order, and the same conclusion has been obtained. Here we apply this gaussian expansion method to the bosonic version of the IIB matrix model, where Monte Carlo results are available, and demonstrate the *convergence* of the method by explicit calculations up to the 7th order. More generally we study matrix models obtained from dimensional reduction of SU(N) Yang-Mills theory in D dimensions, where the $D = 10$ case corresponds to the bosonic IIB matrix model. Convergence becomes faster as D increases, and for $D \gtrsim 10$ it is already achieved at the 3rd order.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua, Matrix Models.

Contents

1. Introduction

It is well appreciated that perturbation theory has a limited range of applicability, although its importance in theoretical physics can never be overstated. First of all, perturbative expansions in most cases yield merely an asymptotic series, which starts to diverge at some finite order. There are also situations in which the expansion parameter is too large to make perturbative calculations reliable or even meaningful.

Superstring theory provides such a situation where nonperturbative effects are considered to be extremely important. From this point of view, the recent proposals for its nonperturbative formulations should be considered as a substantial progress. For instance the IIB matrix model [1] is conjectured to be a nonperturbative definition of type-IIB superstring theory in ten dimensions, and it is speculated to explain the dynamical origin of the Standard Model including the space-time dimensionality, the gauge group and the number of generations. Much effort has been made to understand the dynamics of this model. In particular, Monte Carlo simulations may be as useful as in lattice gauge theories, and indeed various simplified versions of the IIB matrix model have been successfully studied [2, 3]. However, it is not straightforward to extend these studies to the IIB matrix model, where the *complex* fermion determinant causes the notorious complex-action problem. In fact it is speculated that the phase of the fermion determinant plays a crucial role in the dynamical generation of four-dimensional space-time [4]. Recently a new Monte Carlo technique [5] is developed to include the effect of the phase. Preliminary results concerning the space-time dimensionality are encouraging, but it remains to be seen whether definite conclusions can be reached along this line. In any case it would be certainly desirable to develop an alternative method which enables a nonperturbative access to the dynamics of the model.

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In this Letter, we are going to advocate a method to systematically improve the self-consistent harmonic approximation (or the gaussian approximation), which has been widely used in condensed matter physics. (See ref. [6] and references therein.) Indeed such a method was applied to the IIB matrix model up to the 3rd order in ref. [7]. This provided the first analytical evidence for the dynamical generation of *4d space-time*, which is both surprising and encouraging since the model describes superstrings in *10 dimensions*. Stability of this conclusion against higher order corrections has been confirmed recently by explicit 5th order calculations [8]. There the method is also interpreted as an improved Taylor expansion. The gaussian approximation (the leading order of the gaussian expansion) has been applied to random matrix models earlier in ref. [9]. Its application to supersymmetric matrix quantum mechanics (including the Matrix Theory) was advocated by Kabat and Lifschytz [10], and refs. [11] succeeded in revealing interesting blackhole thermodynamics even at the leading order.

Here we apply the method to the bosonic version of the IIB matrix model. More generally we consider the model [12] obtained from dimensional reduction of SU(N) Yang-Mills theory in D dimensions, where the $D = 10$ case corresponds to the bosonic IIB matrix model. The model can also be regarded as a hermitean matrix version of the Eguchi-Kawai model [13], which was proposed as an equivalent description of large-N gauge theory. This equivalence in the case of the present model has been investigated recently [14]. Hence we expect that our results would also have some implications to field theoretical applications of the method.

In the present bosonic model, we perform calculations up to the 7th order, which allows us to convincingly demonstrate the convergence of the method. Although the model is much simpler than the IIB matrix model, it is still nontrivial, and in particular it has not been solved exactly so far. Also the conventional perturbative approach cannot be applied as in the IIB matrix model, since the action does not contain a quadratic term. However, standard Monte Carlo simulation is applicable, and there are some results [15] that can be compared with ours. As another advantage of studying this model, we can perform analytic calculations leaving D as a free parameter. We observe that the convergence is fast in particular for large D. For $D \gtrsim 10$, for instance, a reasonable convergence is already achieved at the 3rd order, which supports the validity of the aforementioned calculations in the IIB matrix model.

We would like to point out that actually the gaussian expansion method has a long history [16, 17], although its application to matrix models is new. Originally it was developed to obtain the energy spectrum in quantum mechanical systems. The most striking aspect of the method is that the results *converge*, although the expansion is *not* based on any small parameter.[1] It is also interesting that the results contain fully nonperturbative effects, although the actual calculations are nothing but the familiar ones in standard perturbation theory. In this approach one typically encounters a situation in which the approximants depend on free parameters. In fact the power of the method comes from the flexibility

[1]There are rigorous proofs of convergence in certain examples [18]. Very recently ref. [19] has shown that complex solutions to the $d(\text{result})/d(\text{unphysical parameter}) = 0$ equation (in fact, the solution with the largest imaginary part) works extremely well.

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to adjust these parameters depending on the order of the expansion. Common wisdom suggests to determine them by 'the principle of minimum sensitivity' [17]. This is usually achieved by solving the 'self-consistency equation'. However, we will encounter some problems with this strategy. Here we propose a novel prescription based on histograms, with which one can naturally obtain the 'best approximation' at each order without such problems. We hope that our histogram prescription is useful in other applications as well.

2. Dimensionally reduced Yang-Mills integrals

The model we study in this work is defined by the partition function

$$Z = \int \mathrm{d}A \, \mathrm{e}^{-S} , \tag{2.1}$$

$$S = -\frac{1}{4} N\beta \sum_{\mu,\nu} \mathrm{tr}[A_\mu, A_\nu]^2 , \tag{2.2}$$

where A_μ ($\mu = 1, \ldots, D$) are $N \times N$ hermitean matrices. The integration measure $\mathrm{d}A$ is defined by $\mathrm{d}A = \prod_{a=1}^{N^2-1} \prod_{\mu=1}^{D} dA_\mu^a/\sqrt{2\pi}$, where A_μ^a are the coefficients in the expansion $A_\mu = \sum_a A_\mu^a T^a$ with respect to the SU(N) generators T^a normalized as $\mathrm{tr}(T^a T^b) = (1/2)\delta^{ab}$. The model can be regarded as the zero-volume limit of SU(N) Yang-Mills theory in D dimensions. As a result of this limit, we can actually absorb β in (2.2) by rescaling $A_\mu \mapsto \beta^{-1/4} A_\mu$. Thus β is not so much a coupling constant as a scale parameter, which we set to $\beta = 1$ without loss of generality. The partition function was conjectured [12] and proved [20] to be finite for $N > D/(D-2)$. Note in particular that the model is ill-defined for $D \leq 2$. A systematic $1/D$ expansion was formulated in [15]. In particular the absence of SO(D) breaking was shown to all orders of the $1/D$ expansion and this conclusion was also confirmed by Monte Carlo simulations [15] for various $D = 3, 4, 6, \ldots, 20$. The gaussian expansion method is also capable of addressing such an issue [7], but here we assume the absence of the SSB in order to focus on the convergence of the method itself. In ref. [21] Polyakov line and Wilson loop were calculated by the (next-)leading gaussian approximation, and Monte Carlo results [3] were reproduced qualitatively. An equally successful result was obtained in the supersymmetric case [22].

3. The gaussian expansion method

The gaussian action which has both SU(N) and SO(D) symmetries can be written as

$$S_0 = \frac{N}{v} \frac{\sqrt{D}}{2} \sum_{\mu=1}^{D} \mathrm{tr}\,(A_\mu)^2 , \tag{3.1}$$

where v is a positive real parameter. (The constant factor $\sqrt{D}/2$ is introduced for convenience.) Then we rewrite the partition function (2.1) as

$$Z = Z_0 \langle \mathrm{e}^{-(S-S_0)} \rangle_0 , \tag{3.2}$$

$$Z_0 = \int \mathrm{d}A \mathrm{e}^{-S_0} , \tag{3.3}$$

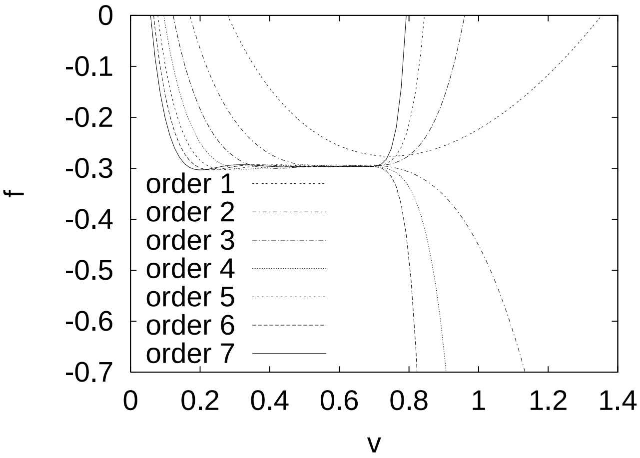

Figure 1: The truncated free energy density f is plotted as a function of v for $D = 10$. Each curve corresponds to order $1, 2, \ldots, 7$. Formation of a plateau is clearly seen.

where $\langle \cdot \rangle_0$ is a VEV with respect to the partition function Z_0. From (3.2) it follows that the free energy $F = -\ln Z$ can be expanded as

$$F = \sum_{k=0}^{\infty} F_k \, ; \qquad F_0 \equiv -\ln Z_0 \, ,$$
$$F_k \equiv -\frac{(-1)^k}{k!} \langle (S - S_0)^k \rangle_{\mathrm{C},0} \quad (\text{for } k \geq 1) \, , \tag{3.4}$$

where the suffix 'C' in $\langle \cdot \rangle_{\mathrm{C},0}$ means that the connected part is taken. Each term F_k in the above expansion can be calculated conveniently by using Feynman diagrams. In actual calculations we have to truncate the infinite series (3.4) at some finite order. In ref. [8], Schwinger-Dyson equations were used to reduce the number of diagrams considerably. Using this technique, we were able to proceed up to the 7th order in the present model with reasonable efforts (e.g., we evaluated 21 eight-loop diagrams.).

Once we truncate the expansion (3.4), the calculated free energy depends on the parameter v introduced in the gaussian action (3.1). In the large-N limit [2] we compute the 'free energy density' defined by

$$f = \lim_{N \to \infty} \left\{ \frac{1}{D(N^2 - 1)} F - \frac{1}{2} \ln \left(\sqrt{\frac{D}{2}} N \right) \right\} . \tag{3.5}$$

The second term is subtracted in order to make the quantity finite. (In particular $f = -1/4$ for $D = \infty$.) In figure 1 we plot the result at each order as a function of v for $D = 10$. Formation of a plateau is clearly seen. We may understand this phenomenon as a reflection of the fact that the formal expansion (3.4), if *not* truncated, should not depend on v. According to this interpretation, the height of the plateau observed for the truncated free

[2] This simply amounts to considering only the planar diagrams. Although it reduces our effort considerably, the method itself is applicable to arbitrary finite N.

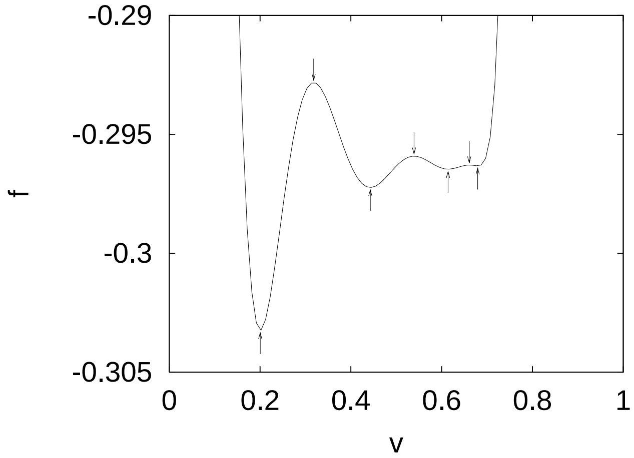

Figure 2: The zoom up of figure 1 for order 7. The arrows indicate the position of the extrema, each of which corresponds to a solution of the 'self-consistency equation'.

energy is expected to provide a good approximation. This is the philosophy behind 'the principle of minimum sensitivity', which is known in a more general context [17].

Next we need to specify a prescription to obtain a concrete value which approximates the free energy at each order. In refs. [7, 8] the free parameters in the gaussian action were determined such that the truncated free energy becomes stationary. This amounts to solving the 'self-consistency equation'

$$\frac{\partial}{\partial v}\left(\sum_{k=0}^{n} F_k\right) = 0\,. \tag{3.6}$$

D	*order* (n)						
	1	2	3	4	5	6	7
3	1	0	1	0	1	0	1
4	1	0	1	0	1	0	1
5	1	0	1	0	1	2	5
6	1	0	3	2	5	4	7
7	1	2	3	4	5	6	7
$\vdots$	1	2	3	4	5	6	7
∞	1	2	3	4	5	6	7

Table 1: The number of solutions to the 'self-consistency equation' (3.6) for each D.

In table 1 we list the number of solutions to the 'self-consistency equation' for each D, which already reveals some problems. For example, for $D = 10$ we find that the number of solutions increases as we go to higher orders, and we have to decide which one to choose. A similar ambiguity was encountered in the IIB matrix model [7, 8], where the solution that gives the smallest free energy has been chosen. While this prescription seems reasonable at relatively small orders, we have problems at higher orders, in particular when we are discussing the convergence. In figure 2 we zoom up the plateau seen at the order 7 in figure 1. We observe some oscillations, whose amplitude becomes larger towards the left edge of the plateau. If we used the prescription described above in the present situation, we would clearly underestimate the height of the plateau. This will be even more problematic at higher orders since the dip at the left edge seems to become deeper as the order increases. It may also happen in general that the truncated free energy acquires a global minimum which has nothing to do with the plateau formation.

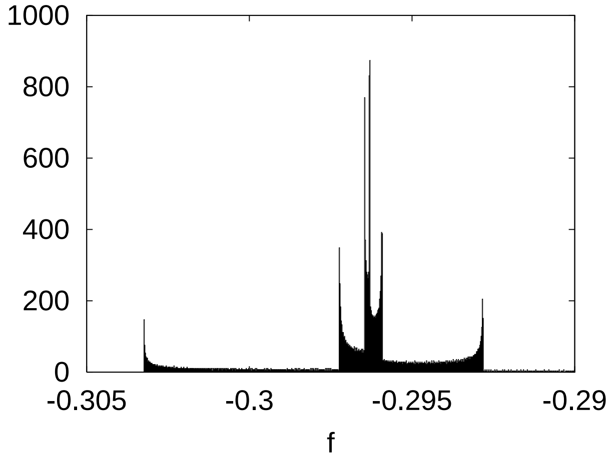

Figure 3: The histogram of the truncated free energy density f for $D = 10$ at order 7. The bin size is chosen to be very small (0.00002).

From table 1 one also finds that there are cases where the 'self-consistency equation' has no solutions. (This was encountered in the study of the IIB matrix model at the orders 2 and 4 [7, 8].) For instance there is no solution at the orders 2,4,6 for $D = 4$. As we show in the Appendix, a well-defined plateau is developing even in this case. The question is how to extract the height of the plateau in such cases.

4. The histogram prescription

All these problems arise from trying to *determine* the free parameters by local information such as 'self-consistency equations'. In fact it is more natural to extract the height of the plateau *directly*. As a first step, let us make a histogram of the truncated free energy calculated at points which are uniformly distributed in the parameter space of the gaussian action. Figure 3 shows the histogram for $D = 10$ at order 7. Whenever we make histograms in what follows, the range of v is restricted to $0.01 \leq v \leq 1.4$, and the vertical axis is normalized such that the integration gives unity. In figure 3 the bin size is chosen to be very small, and as a result we obtain many peaks, each of which corresponds to a solution of the 'self-consistency equation' (Actually the highest peak contains the two solutions which lie at the right edge of the plateau in figure 2.). Hence we are back in the previous situation, and we cannot reasonably extract the height of the plateau. In order to identify the plateau, we clearly need some kind of *coarse graining*. Let us note here that the bin size of the histogram represents the precision at which we search for the plateau. Therefore, we increase the bin size gradually until a single peak becomes 'dominant'. As a criterion of the dominance, we require the highest bin to be more than twice as high as the second highest one. We further require the dominance to persist for larger bin size. The resulting histogram is shown in figure 4.

Figure 5 shows the histograms obtained in this way at each order for $D = 10$. The peak becomes sharper and sharper as we go to higher order. Thus the histogram serves

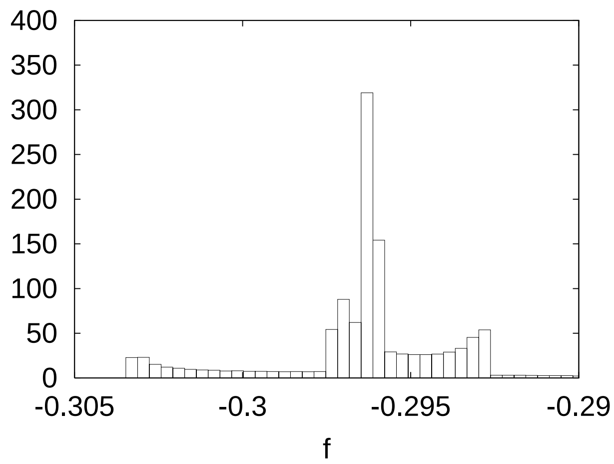

Figure 4: The histogram of the truncated free energy density f for $D = 10$ at order 7. The bin size (0.00035) is chosen such that the highest bin is more than twice as high as the second highest one.

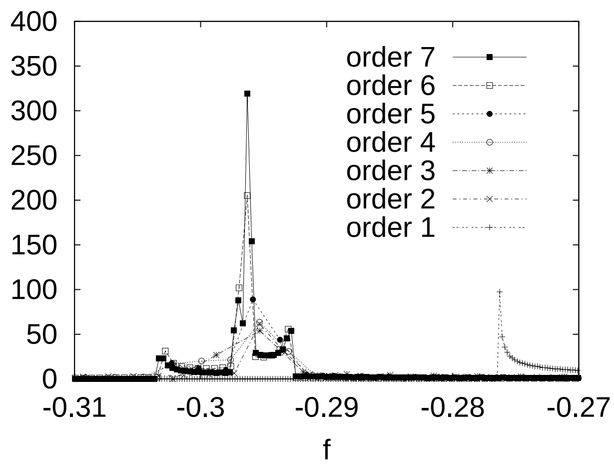

Figure 5: The histogram of the truncated free energy density f for $D = 10$ at order $1, 2, \ldots, 7$. The bin size is chosen carefully as described in the text at each order. The growth of the peak is clearly observed.

as a clear indicator of the plateau formation. Moreover, now we can obtain the 'best approximation' at each order from the position of the peak, while the bin size (chosen by the above criterion) roughly represents the theoretical uncertainty of the method. In the appendix, we present a similar analysis for $D = 4$. In this case the 'self-consistency equation' has no solutions at orders 2,4,6 as mentioned above. However, the histogram prescription is as successful as in $D = 10$.

In figure 6, we plot the approximated value of the free energy for various D obtained at each order by our histogram prescription. The 'error bars' represent a rough estimate for the theoretical uncertainty explained above.[3] (Remark: Unlike usual error bars, they do

[3]We did not put error bars to the order 1 results. This is because our criterion for the dominance of

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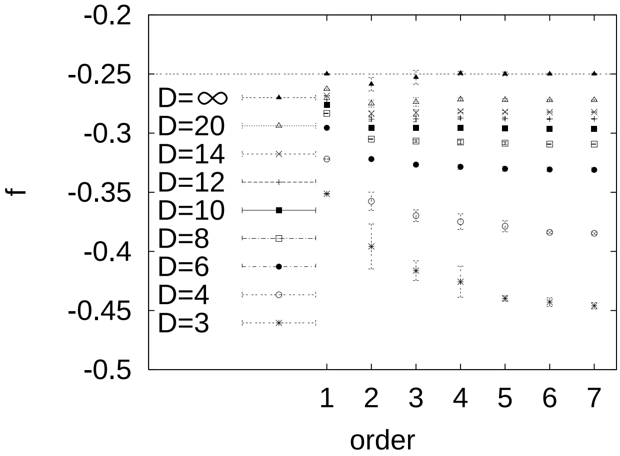

Figure 6: Free energy density f obtained by the histogram prescription at order $1, 2, \ldots, 7$ for various D. The horizontal line represents the exact result $f = -1/4$ for $D = \infty$.

not represent an estimate for possible discrepancies from the correct value. For the latter, one should also consider the changes of the results as the order is increased.) We observe a clear convergence for each D. The convergence becomes fast in particular for large D.

5. Observables

Similarly to the free energy (3.4), the expectation value of an operator $\mathcal{O}$ is expanded as

$$\langle \mathcal{O} \rangle = \sum_{k=0}^{\infty} O_k \, ; \qquad O_0 \equiv \langle \mathcal{O} \rangle_0 \, ,$$
$$O_k \equiv \frac{(-1)^k}{k!} \Big\langle (S - S_0)^k \mathcal{O} \Big\rangle_{\mathrm{C},0} \qquad (\text{for } k \geq 1) \, . \tag{5.1}$$

As a fundamental object in this model [15], we consider

$$\mathcal{O} = \frac{1}{N} \sum_{\mu=1}^{D} \mathrm{tr}\, (A_\mu)^2 \times \sqrt{\frac{2}{D}} \, . \tag{5.2}$$

The normalization is chosen such that $\langle \mathcal{O} \rangle = 1$ for $D = \infty$ in the large-N limit [15]. We calculate $\langle \mathcal{O} \rangle$ by the series expansion (5.1) truncated at some order. In figure 7 we plot the result at each order as a function of v for $D = 10$.

In refs. [7, 8], the truncated expectation value was evaluated at the same value of v as the one used to evaluate the free energy. Note, however, that the region of the parameter v where the plateau develops for the expectation value is slightly shifted from the corresponding region for the free energy. It is therefore safer to extract an approximated value for the observable by repeating the same procedure as we did for the free energy.

a single bin is satisfied even for an infinitesimal bin size. Thus at the order 1, the histogram prescription does nothing more than just solving the 'self-consistency equation'.

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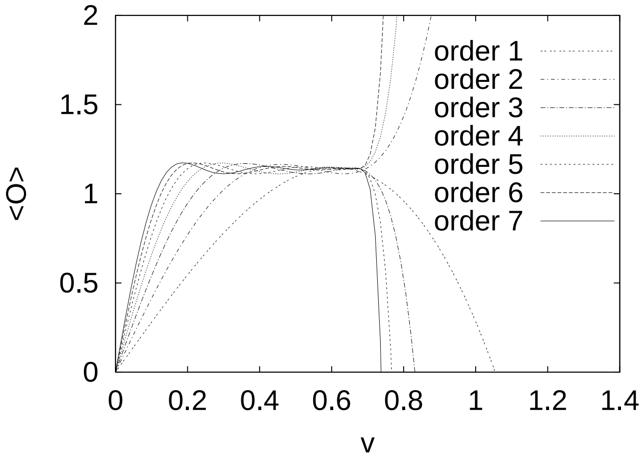

Figure 7: The truncated expectation value $\langle \mathcal{O} \rangle$ is plotted as a function of v for $D = 10$. Each curve corresponds to order $1, 2, \ldots, 7$. Formation of a plateau is clearly seen.

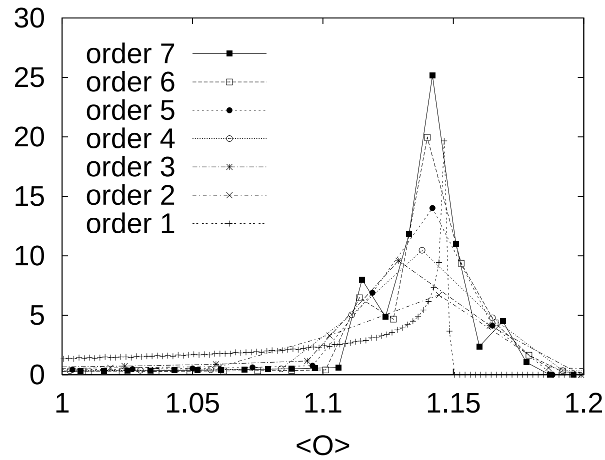

Figure 8: The histogram of the truncated expectation value $\langle \mathcal{O} \rangle$ for $D = 10$ at order $1, 2, \ldots, 7$. The bin size is chosen carefully as described in the text at each order. The growth of the peak is clearly observed.

Figure 8 shows the results of the histogram prescription at each order. The growth of the peak is clearly observed, which confirms the plateau formation. Figure 9 summarizes the final results for the observables, where the 'error bars' again represent the theoretical uncertainty. Comparison with the Monte Carlo data [15] confirms that our results do converge to the correct values. For $D \gtrsim 10$ in particular, the convergence is already achieved at order 3.

It was previously noted that the leading gaussian approximation becomes 'exact' at $D = \infty$ [21]. The precise statement is that if we evaluate the free energy density and the observable (truncated at the 1st order) at the value of v which solves the 1st order 'self-consistency equation' *for the free energy*, they agree with the exact results at $D = \infty$. Although this can be naturally understood from the viewpoint of the $1/D$ expansion [15], it

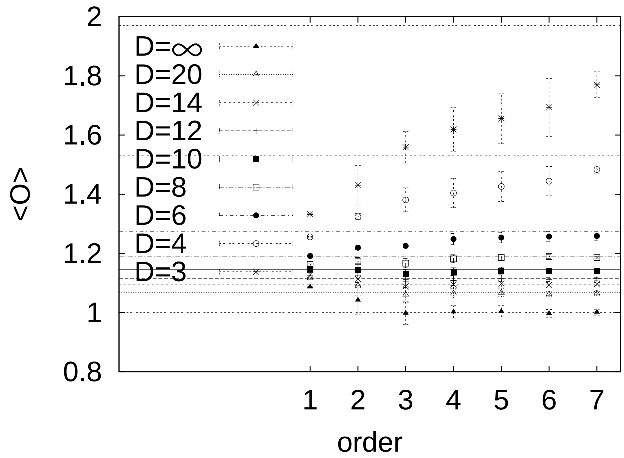

Figure 9: The expectation value $\langle \mathcal{O} \rangle$ obtained by the histogram prescription at order $1, 2, \ldots, 7$ for various D. The horizontal lines represent the Monte Carlo results for each $D < \infty$ and the exact result $\langle \mathcal{O} \rangle = 1$ for $D = \infty$.

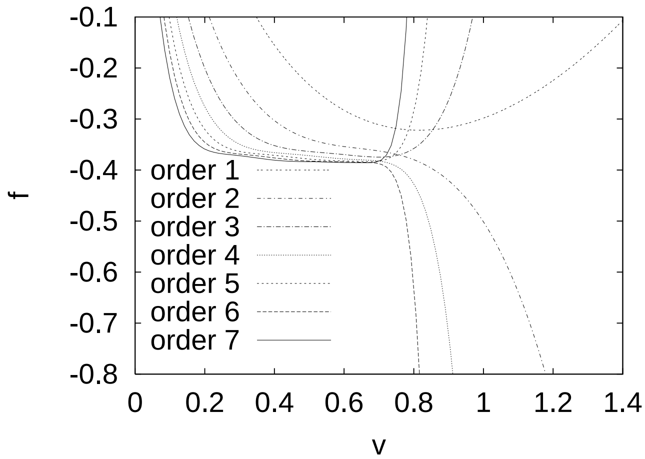

Figure 10: The truncated free energy density f is plotted as a function of v for $D = 4$. Each curve corresponds to order $1, 2, \ldots, 7$. Formation of a plateau is clearly seen.

looks rather accidental in the present framework. If we evaluate the truncated observable at v which solves the 'self-consistency equation' *for the observable*, we obtain 1.08866 instead of 1. Also there is no plateau yet at order 1. Figures 6 and 9 show that the convergence of the method as it stands is achieved at order 3 (but not yet at order 1) even for $D = \infty$.

In this regard, it should be emphasized that the gaussian expansion is *different* from the $1/D$ expansion. Each term in the expansions for the free energy density and the observable is *of order 1* with respect to $1/D$ for generic v. Still the height of the plateau converges due to nontrivial cancellations among diagrams. Note also that the gaussian expansion can be applied to the supersymmetric case (including the IIB matrix model), while the $1/D$ expansion cannot be formulated there.

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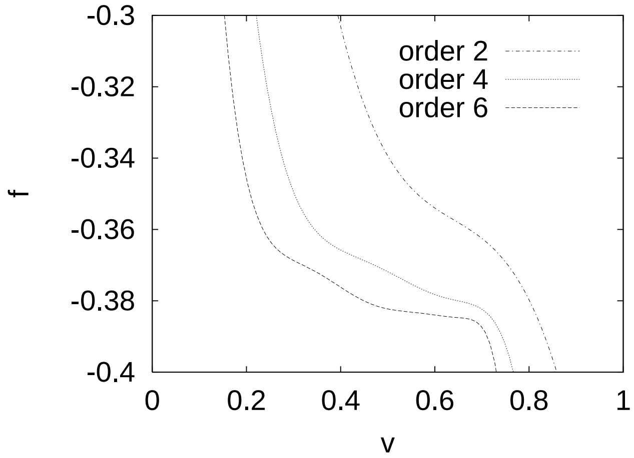

Figure 11: The zoom up of figure 10 for order 2,4,6. There is no extremum point in the curves.

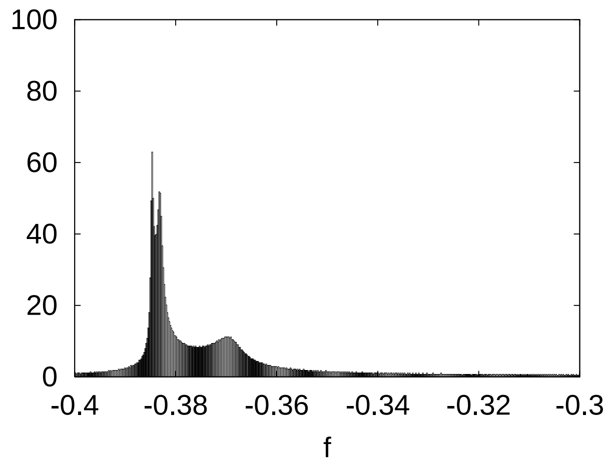

Figure 12: The histogram of the truncated free energy density f for $D = 4$ at order 6. The bin size is chosen to be very small (0.0002).

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6. Summary and discussions

The most important conclusion of our study is that the gaussian expansion method converges in the dimensionally reduced Yang-Mills integrals. In particular the convergence is achieved already at order 3 for $D = 10$, which corresponds to the bosonic IIB matrix model. This suggests the particular usefulness of the method in studying the IIB matrix model. In demonstrating the convergence we encountered problems associated with the commonly used prescription based on 'self-consistency equations'. We solved these problems by extracting the height of the plateau directly using a novel prescription based on histograms.

The histogram prescription is expected to be particularly useful in the case where the parameter space of the gaussian action becomes multi-dimensional. If there are more than two parameters, it is already difficult to visualize the plateau formation as we did in

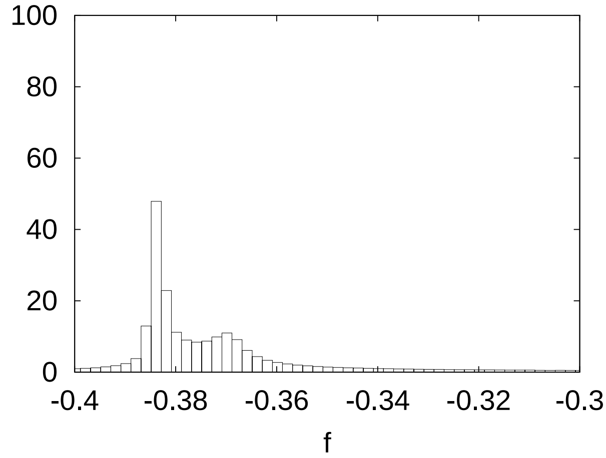

Figure 13: The histogram of the truncated free energy density f for $D = 4$ at order 6. The bin size (0.002) is chosen such that the highest bin is more than twice as high as the second highest one.

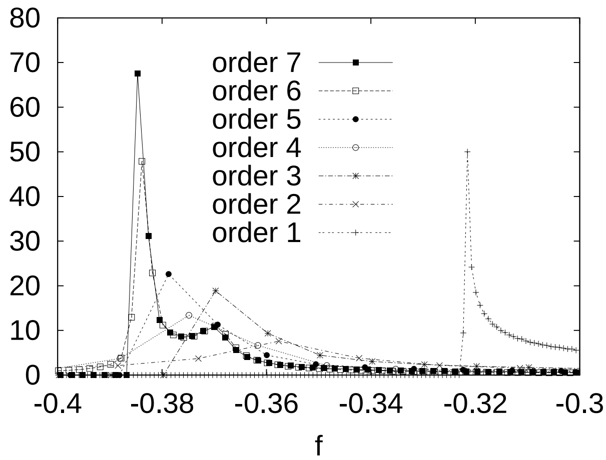

Figure 14: The histogram of the truncated free energy density f for $D = 4$ at order $1, 2, \ldots, 7$. The bin size is chosen carefully as described in the text at each order. The growth of the peak is clearly observed.

figures 1, 7 and 10. When we study the SSB of SO(10) symmetry in the IIB matrix model, we have to introduce 10 real and 120 complex parameters [7]. Here we will need some kind of important sampling technique like Monte Carlo simulations in order to explore such a huge parameter space. In the previous works [7, 8] based on 'self-consistency equations', there were problems like the ones we saw in this work. At order 2 and 4 there were no solution to the 'self-consistency equations', and in the other cases the solution was not unique. We are currently re-investigating the IIB matrix model by using the histogram prescription with the above extension [23].

Finally we would like to emphasize that the dimensionally reduced model we studied in this paper is a system of *infinitely many degrees of freedom*, since we have taken the size of the matrices A_μ to be infinite. In this regard, let us recall that the model has also connections to SU(N) gauge theories in D-dimensional space time in the large-N limit [13]. The dimensional reduction amounts to reducing the base space of the SU(N) gauge theories to a point, but the original space-time degrees of freedom are still somehow encoded in the internal degrees of freedom. This is possible precisely because there are infinitely many internal degrees of freedom after taking the large-N limit. From this point of view the convergence of the gaussian expansion method in the present model is interesting since the examples for which the convergence is shown in the literature have been restricted to simple quantum-mechanical systems. Although higher order calculations in field-theoretical applications would be technically more involved due to the necessity of the renormalization procedure (See, however, ref. [24]), we hope that our results open up a new perspective in that direction as well.

Acknowledgments

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A. The situation in $D = 4$: plateau without extrema

In this appendix, we present our results for $D = 4$. As we see from figure 10, a well-defined plateau is clearly developing. However, if we zoom up the curves for order 2,4,6 (figure 11), we find that there are no extrema, in agreement with the absence of solutions to the 'self-consistency equation' (See table 1). We will see that the histogram prescription works even in this case.

Figures 12 and 13 show the histograms at order 6 before and after 'coarse-graining'. As we increase the bin size gradually, a single bin becomes dominant (See section 4 for the precise criterion.). Thus the histogram prescription allows us to obtain explicit results at each order including orders 2,4,6, where the 'self-consistency equation' has no solutions. Figure 14 shows the results at orders $1, \ldots, 7$. The growth of the peak is clearly observed, which confirms the plateau formation. Moreover, the results obtained in this way show a clear convergence as we have seen in figure 6.

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